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Stability of circular Couette flow with variable inner cylinder speed

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Energy-stability theory is employed to study the finite-amplitude stability of a viscous incompressible fluid occupying the space between a pair of concentric cylinders when the inner-cylinder angular velocity varies linearly with time. For the case with a fixed outer cylinder and increasing inner-cylinder speed, we find an enhancement of stability, consistent with a linear-theory result due to Eagles. When the inner-cylinder speed decreases, we find an initially decreased stability bound, indicating the possibility of hysteresis, while, if the inner cylinder is allowed to reverse direction and linearly increase in speed, we find significant stability enhancement.

1. Introduction

The stability of flows between concentric rotating cylinders has been an area of vigorous study since the original work of Taylor (1923). The great bulk of the work done to date has been concerned with the instability of steady Couette flow. The recent review article by Di Prima & Swinney (1981) gives an excellent account of the theoretical and experimental research on this problem.

Recent interest in the stability of unsteady flows in general has drawn a significant amount of attention to various types of unsteady Couette flow. Chen & Kirchner (1971) and Liu & Chen (1973) examine Couette flow with an impulsively started inner cylinder, while Hall (1975), Riley & Laurence (1976) and Tustaniwskyj & Carmi (1980) treat the case with a modulated inner-cylinder speed. Chen, Liu & Skok (1973) assume the inner cylinder is taken from a state of rest to its final angular velocity at a constant acceleration. Seminara & Hall (1975) and Eagles (1977) analyse cases whose basic states are assumed to be slowly varying, so that the WKBJ method could be applied; the former assume a slowly varying azimuthal pressure gradient, while the latter assumes that the inner-cylinder angular velocity is a slowly varying quantity. Neitzel (1982) considers Couette flow initiated by impulsively stopping the outer cylinder, the entire system having been in an initial state of rigid-body rotation, and Neitzel & Davis (1980) examine the related problem of spin decay within a single cylinder.

With the exception of the work of Tustaniwskyj & Carmi (1980), Neitzel & Davis (1980) and Neitzel (1982), all of the aforementioned research has relied on linear stability theory to obtain results. For cases with time-periodic basic states, linear theory may be applied in a relatively straightforward fashion since Floquet theory guarantees the existence of an exponential time factor and hence a well-defined growth rate. For time-aperiodic basic states, however, Floquet theory no longer applies. In these cases, one must either restrict one's attention to slowly varying flows to allow use of the WKBJ approximation, making further assumptions regarding a stability criterion, or base a stability decision upon some arbitrary quantity such as a disturbance amplification factor. The three investigations noted above employ energy-stability theory under various stability criteria to treat their respective problems. A strong-stability criterion requires that disturbances to the basic state decay to guarantee stability while a marginal-stability criterion only requires that disturbances be smaller in magnitude than their initial values. The energy-theory formulation yields a set of linear governing equations that are only parametrically dependent on time while restricting neither disturbance amplitude nor rate of change of the basic state.

The problem of interest here is related to that of Eagles (1977). Consider a pair of infinitely long concentric circular cylinders of radii a and $b > a$ whose gap is filled with a viscous, incompressible fluid. We assume that the initial state of the system is steady Couette flow characterized by a Reynolds number $R_0 = \Omega(b-a)^2/\nu$, where Ω is the angular velocity of the inner cylinder (the outer cylinder is assumed fixed) and ν is the kinematic viscosity of the fluid. At time $t = 0$, the inner-cylinder angular velocity $\omega(t)$ begins to increase or decrease linearly with time. The problem of Eagles (1977) varies slightly in that his initial condition is somewhat different (he assumes a flow starting from rest at $t = -\infty$) and he also treats cylinder speeds varying quadratically in time. Eagles assumes that the inner-cylinder angular velocity is *slowly varying* in time so that he can use linear stability theory coupled with the WKBJ approximation. The present analysis employs energy-stability theory under a strong-stability assumption so that neither the rate of change of the cylinder speed nor the disturbance amplitude is limited. Strong stability guarantees exponential decay of disturbances as long as points on an experimental trajectory are below the critical curve in the Reynolds-number–time plane.

We examine the stability of this flow for two different values of cylinder radius ratio $\eta = a/b$ equal to 0·5 and 0·95, and several accelerations, both small and large. For $\eta = 0·5$, the results of energy theory and linear-theory calculations for *steady* Couette flow are quite close (see Di Prima & Swinney 1981). We find, for the unsteady problem with $\eta = 0·5$ and ω increasing, that the flow is stable at instantaneous Reynolds numbers significantly above the linear-theory critical value for steady Couette flow. We also find that there exists a starting condition that maximizes this enhancement of stability. Cases for which the inner-cylinder speed decreases, changes direction and then increases in the opposite direction show an initial reduction in critical Reynolds number over the steady-state value (indicating a possible hysteresis) followed by an enhancement after the direction change.

2. Basic state

We assume a viscous, incompressible fluid occupies the gap between a pair of infinitely long, concentric circular cylinders of radii a and b. The flow is assumed to be in an initial state of steady Couette flow with the outer cylinder fixed and the inner cylinder rotating at constant angular velocity Ω. The Reynolds number characterizing this flow is $R_0 = \Omega(b-a)^2/\nu$. At time $t = 0$, the inner-cylinder angular velocity begins to vary according to the relation $\omega(t) = \Omega(1+\tilde{A}t)$, where $\tilde{A}$ is a positive or negative constant of unrestricted magnitude.

Let (r, θ, z) be the usual cylindrical co-ordinates with corresponding velocity components (U, V, W). We seek a pure-swirl basic state of the form

$$\mathbf{U} = (0, V(r,t), 0).$$

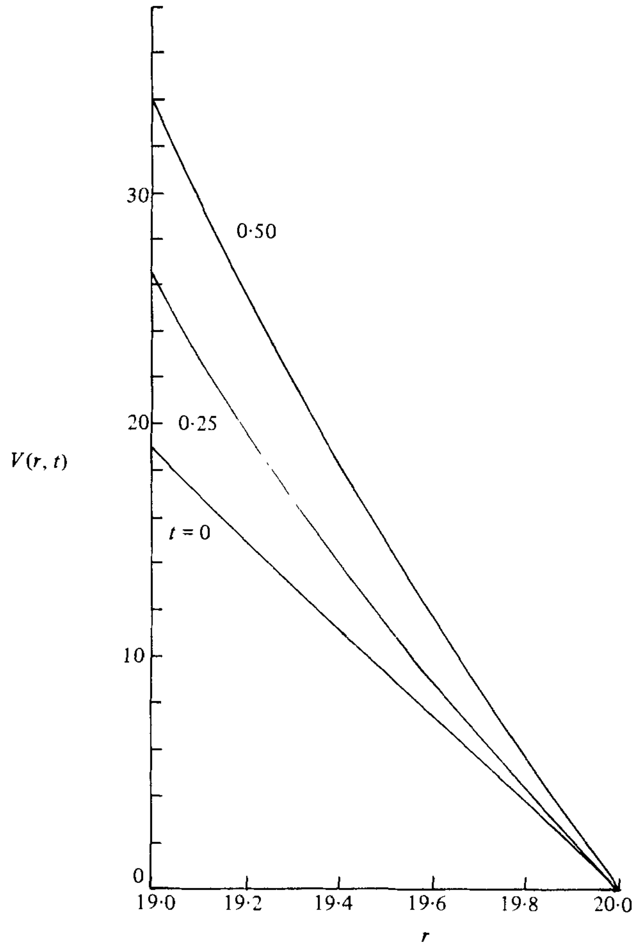

FIGURE 1. Velocity profiles for $\eta = 0{\cdot}95$ and $A = 1{\cdot}6$ for various times.

Under the above assumptions, the initial-boundary-value problem governing $V(r,t)$ is given by

$$V_t = \nu(V_{rr} + r^{-1}V_r - r^{-2}V), \tag{2.1}$$

$$V(a,t) = a\Omega(1+\tilde{A}t), \tag{2.2}$$

$$V(b,t) = 0, \tag{2.3}$$

$$V(r,0) = \frac{-a^2\Omega}{b^2-a^2}r + \frac{a^2b^2\Omega}{b^2-a^2}\frac{1}{r}. \tag{2.4}$$

An analytical solution to (2.1)–(2.4) may be obtained in a straightforward fashion by using Laplace transforms and Duhammel's Principle. This solution may be found in the appendix and was used to compute basic-state values required by the computations which follow. Values at various radii for selected times and radius ratios were computed using an implicit finite-difference technique to check these results. Figures 1 and 2 show velocity profiles at various times for cases of positive and

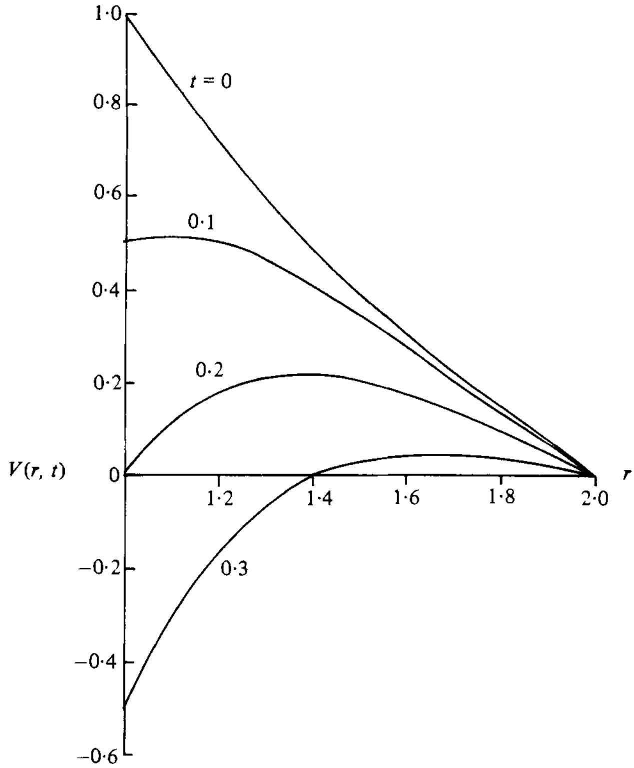

FIGURE 2. Velocity profiles for $\eta = 0{\cdot}5$ and $A = -5{\cdot}0$ for various times.

negative acceleration constant. The results were non-dimensionalized using the following scales:

$$\text{length} \rightarrow b-a,$$

$$\text{time} \rightarrow (b-a)^2/\nu,$$

$$\text{velocity} \rightarrow (b-a)\Omega.$$

These profiles might lead one to conclude for cases with $A > 0$ (figure 1, A is the non-dimensionalized $\tilde{A}$) that one will eventually reach a time beyond which the flow will never again be stable, while for cases with $A < 0$ (figure 2) that there are perhaps intermediate times for which disturbances to the basic state must decay. This conclusion is borne out by the results of the stability computations.

3. Stability analysis

As mentioned in § 1, Eagles allows inner-cylinder speed variations that are functions of a 'slow' time $\tau = \epsilon t$, where $\epsilon \ll 1$. In this case, the basic state varies on a timescale much longer than the diffusive timescale on which disturbances develop. For basic states whose time dependence arises from an *impulsive* change in boundary conditions, this approach fails since both the basic state and the disturbances to it evolve on the same diffusive timescale. Neitzel & Davis (1980) and Neitzel (1982) have employed energy-stability theory to treat such flows within rotating cylinders. In applying energy theory to the present problem, one can treat not only the slowly varying cases of Eagles, but also cases of rapidly changing cylinder speed, which are out of the range of applicability of linear stability theory.

The energy identity is derived in the usual fashion (Serrin 1959) by taking the inner product of the disturbance velocity vector $\mathbf{u} = (u, v, w)$ with the nonlinear disturbance equations and integrating over a volume $\mathscr{V}$ of a cell defined by

$$\mathscr{V} = \left\{(r, \theta, z) \,|\, \frac{\eta}{1-\eta} \leqslant r \leqslant \frac{1}{1-\eta}, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant z \leqslant Z\right\},$$

where we have assumed axial periodicity with period Z in anticipation of Taylor-vortex-type disturbances. The result is

$$\frac{dE}{dt} = RI - D, \tag{3.1}$$

where

$$E \equiv \tfrac{1}{2}\langle \mathbf{u}\,.\,\mathbf{u}\rangle, \quad I \equiv \langle uv(V/r - V_r\rangle, \quad D \equiv \langle \nabla\mathbf{u} : \nabla\mathbf{u}\rangle.$$

R is the Reynolds number defined by

$$R \equiv \frac{\Omega(b-a)^2}{\nu}$$

and $\langle\;\rangle$ denotes integration over $\mathscr{V}$. The flow is said to be strongly stable if $R < R_{\mathrm{E}}$, where

$$\frac{1}{R_{\mathrm{E}}} = \max_{S} \frac{I}{D}, \tag{3.2}$$

and

$$S = \left\{\mathbf{u} | \mathbf{u} \in C^2, \quad \mathbf{u} = 0 \quad \text{for} \quad r = \frac{\eta}{1-\eta}, \; \frac{1}{1-\eta}, \right.$$
$$\left. \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \text{ periodic in } z \text{ with period } Z\right\}$$

is the usual set of kinematically admissible functions. Serrin (1959) has shown that, for $R < R_{\mathrm{E}}$, the integrated disturbance kinetic energy E obeys the inequality $E^{-1}(dE/dt) < 0$, implying energy decay. Hence, $R < R_{\mathrm{E}}$ is a sufficient condition for stability to disturbances of arbitrary amplitude.

The Euler–Lagrange equations corresponding to the variational problem (3.2) are

$$\tfrac{1}{2}\tilde{R}\Psi v + \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} - \frac{1}{r^2}u - \frac{2}{r_2}v - \pi_r = 0, \tag{3.3a}$$

$$\tfrac{1}{2}\tilde{R}\Psi u + \frac{1}{r}(rv_r)_r + \frac{1}{r^2}v_{\theta\theta} + v_{zz} - \frac{1}{r^2}v + \frac{2}{r^2}u - \frac{1}{r}\pi_\theta = 0, \tag{3.3b}$$

$$\frac{1}{r}(rw_r)_r + \frac{1}{r^2}w_{\theta\theta} + w_{zz} - \pi_z = 0, \tag{3.3c}$$

$$\frac{1}{r}(ru)_r + \frac{1}{r}v_\theta + w_z = 0, \tag{3.3d}$$

where

$$\Psi(r, t) = r^{-1}V(r, t) - V_r(r, t)$$

represents the contribution of the basic state. $\tilde{R}$ and π are Lagrange multipliers, with $\tilde{R}$ having the significance that

$$R_{\mathrm{E}} = \min_{T} \tilde{R},$$

where T is identical with S except for the fact that the solenoidal condition on $\mathbf{u}$ has been relaxed. The appropriate boundary conditions are

$$\mathbf{u} = 0 \quad \text{at} \quad r = \frac{\eta}{1-\eta}, \quad \frac{1}{1-\eta}. \tag{3.4}$$

Equations (3.3*a–d*) are linear, although no assumptions restricting disturbance amplitude have been made, and the dependence on time is now parametric through the quantity $\Psi(r, t)$. This latter fact eliminates the necessity of making an arbitrary decision regarding the onset of instability based on disturbance amplification as has been done in some linear-theory work on related problems (see e.g. Chen *et al.* 1973).

An interpretation of the results of this type of analysis is given in Homsy (1973) and in Neitzel & Davis (1980) and is omitted here. The theory is capable of providing a lower bound on the onset time of an instability by guaranteeing that before such a time all disturbances must decay. Likewise, in certain circumstances, one may obtain an upper bound on a decay time, i.e. a time after which all disturbances to the basic state must decay. A discussion related directly to the results obtained for the present flow will follow shortly.

4. Solution procedure

We begin our determination of R_E by making the usual assumption that the disturbances are axisymmetric. Strong-stability calculations by Chen & Neitzel (1982) for impulsively initiated Couette flow show that such disturbances have both the earliest bounds on onset times and the latest bounds on decay times, so that even though the present flow is somewhat different, we have some confidence that our assumption is proper. We then assume the disturbances can be decomposed into normal modes as

$$(u, v, \pi) = (\tilde{u}, \tilde{v}, \tilde{p}) \cos kz, \quad w = \tilde{w} \sin kz,$$

where $k = 2\pi/Z$ is the axial wavenumber. Under these assumptions (3.3*a–d*) reduce to (dropping tildes)

$$\tfrac{1}{2}R\Psi v + u'' + \frac{1}{r}u' - \left(k^2 + \frac{1}{r^2}\right)u - p' = 0, \tag{4.1a}$$

$$\tfrac{1}{2}R\Psi u + v'' + \frac{1}{r}v' - \left(k^2 + \frac{1}{r^2}\right)v = 0, \tag{4.1b}$$

$$w'' + \frac{1}{r}w' - k^2 w + kp = 0, \tag{4.1c}$$

$$u' + \frac{u}{r} + kw = 0, \tag{4.1d}$$

where a prime denotes differentiation with respect to r. (The boundary conditions (3.4) remain the same.) The determination of R_E at a fixed time is now accomplished by minimizing the eigenvalues R of the system (4.1) with boundary conditions (3.4) over the space of axial wavenumbers k, i.e.

$$R_\mathrm{E} = \min_k |R|. \tag{4.2}$$

The calculation procedure is to fix t and compute the smallest positive eigenvalue R of the system (4.1) subject to the boundary conditions (3.4) for various axial wavenumbers k in order to determine R_E using (4.2). Eigenvalues were computed using a standard shooting technique with fourth-order Runge–Kutta integration and 200 integration steps. A fixed-step-size integration routine was used so that, for given t, $\Psi(r,t)$ could be evaluated at predetermined values of r and stored. Therefore, each iteration required by the shooting method can use these same values, eliminating the time-consuming evaluation of Bessel functions, which would be necessary for each iteration if a variable-step-size integrator were used. The above procedure produces one point in the (R_E, t)-plane. Time was then varied to produce a curve of R_E versus time for each radius ratio η and acceleration A considered. Points below such curves correspond to strong stability of the unsteady Couette flow under consideration.

5. Results and discussion

Stability results have been obtained for radius ratios $\eta = 0{\cdot}5$ and $\eta = 0{\cdot}95$ for various values of A, both positive and negative. Most of the results are for the case $\eta = 0{\cdot}5$, since it is here that energy and linear limits most nearly coincide for *steady* Couette flow. Results are presented in terms of $R_0 \equiv \Omega(b-a)^2/\nu$, the initial Reynolds number $R_I \equiv \omega(t)(b-a)^2/\nu$, the instantaneous Reynolds number based on $\omega(t)$, and R_S, the energy-theory value for steady Couette flow. Results for the narrow-gap case of $\eta = 0{\cdot}95$ are presented in figures 3–6, while those for the wide-gap case of $\eta = 0{\cdot}5$ are given in figures 7 and 8.

Figure 3 shows the results obtained for $\eta = 0{\cdot}95$, $A = 1{\cdot}6$ (this value of A corresponds to Eagles' $\epsilon = 0{\cdot}4$, a choice that is quite large in the sense of $\epsilon \ll 1$). R_E has been scaled by R_S for ease of interpretation. An experimental trajectory would be represented on this plot by a horizontal line at a starting condition defined by R_0/R_S. The flow is strongly stable as long as a point on this experimental trajectory lies below the curve. Therefore, associated with each starting condition in the range $0 < R_0/R_S < 1$ is a lower bound on onset time defined by the time at which the experimental trajectory intersects the curve of R_E/R_S. As expected, R_E/R_S intersects the ordinate at the value $R_E/R_S = 1$, implying that if one starts above the steady-state value and increases ω, stability can never be guaranteed. For smaller starting values, stability is guaranteed for increasingly longer periods of time with the curve asymptotically approaching zero. This is the anticipated behaviour in light of the assumed form for $\omega(t)$. Since $\omega(t) = \Omega(1+\tilde{A}t)$, if $R_0 = 0$ (implying $\Omega = 0$), then $\omega(t)$ will be zero for all time.

An alternate method of representing these results is to use the time t corresponding to a particular value of R_0/R_S to compute $\omega(t)$ and hence $R_{I,c}/R_S$. Figure 4 shows the results of figure 3 presented in this way. As before, stability is guaranteed for values falling below the curve. Immediately apparent from this figure is the fact that the instantaneous Reynolds number $R_{I,c}$ below which stability is guaranteed, is above the steady-state value R_S. This enhancement of stability is in accord with that shown by the linear-theory, slowly varying approach of Eagles (1977). Perhaps just as interesting is the fact that there exists an optimum starting value R_0 for which the enhancement is maximized. In this case, this occurs for R_0 of about 90 % of the steady-state critical value ($R_S = 4{\cdot}24$ for $\eta = 0{\cdot}95$). Starting at this value will result in stability being guaranteed against disturbances of arbitrary amplitude until R_I is roughly 4 % above R_S. Eagles' 20 % shift in critical Taylor number for linear theory translates to a 10 % shift in Reynolds number for his analysis of infinitesimal disturbances.

Results for the case of $\eta = 0{\cdot}95$, $A = -1{\cdot}6$ (ω decreasing) are given in figures 5 and 6.

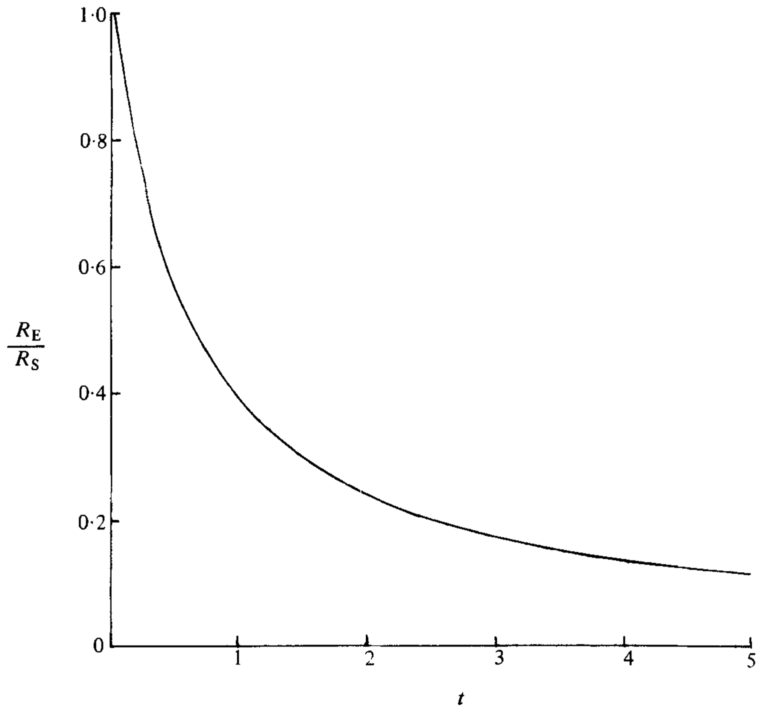

FIGURE 3. R_E/R_S versus time for $\eta = 0{\cdot}95$, $A = 1{\cdot}6$. An experimental trajectory is represented by a horizontal line at a starting condition defined by R_0/R_S.

In figure 5, R_E/R_S is plotted against time. For this choice of the constant A, $|\omega(t)|$ decreases linearly until $t = 0{\cdot}625$ and then begins to increase owing to the fact that ω changes sign at that time. For starting values R_0 less than the steady critical value, stability is guaranteed until some point far beyond the point where the cylinder has reversed direction. Eagles' results for the decreasing case imply that if one starts *below* the steady, linear-theory critical Taylor number and decreases ω linearly, the instantaneous critical Taylor number will be somewhat lower. If one begins with steady Couette flow as assumed here and does this, then this result seems physically unrealistic and the present results show this to be impossible. If one assumes that this starting value was reached by decreasing ω from a state that was *unstable*, then a linear-theory analysis based on a pure-Couette-flow basic state does not apply, since a Taylor vortex constitutes a disturbance of finite amplitude. Our energy-theory approach does not suffer from this defect, since disturbance size is unrestricted. Therefore, when starting at Reynolds numbers that correspond to unstable steady Couette flow, everything other than our prescribed basic state must be considered a disturbance. When the experimental trajectory corresponding to this starting condition enters the region of guaranteed strong stability, *all* disturbances to our basic state (including any remnants of a Taylor vortex) must begin to decay.

Figure 5 shows that if one starts at $R_0/R_S > 12$ then the flow is never guaranteed to be stable. On the other hand, if one begins in the range $1 < R_0/R_S < 12$, then each experimental trajectory will pass through a region of strong stability in which disturbances to the basic state must experience exponential decay. Admittedly, the amount of time spent in this region may not be long enough for the disturbance to

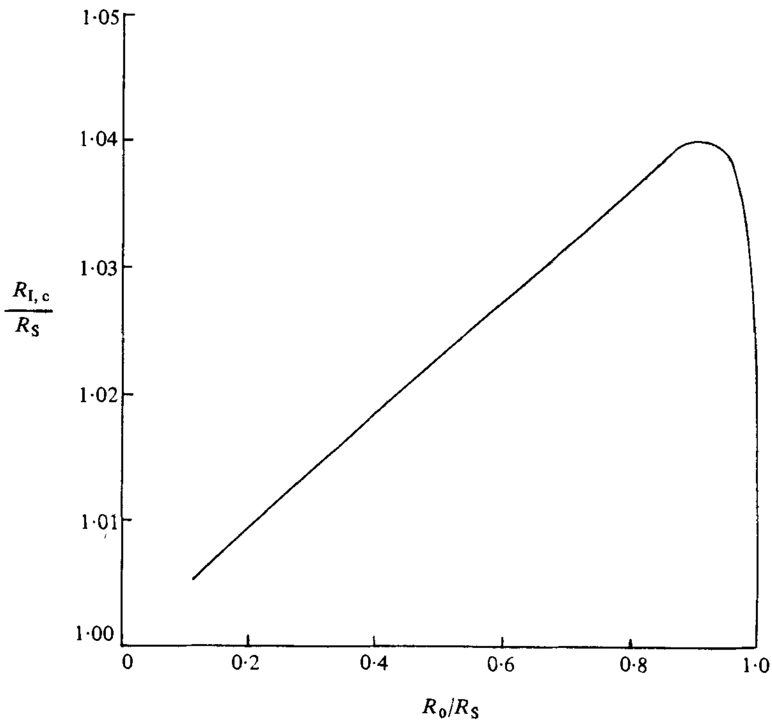

FIGURE 4. $R_{I,c}/R_S$ versus R_0/R_S for $\eta = 0{\cdot}95$, $A = 1{\cdot}6$.

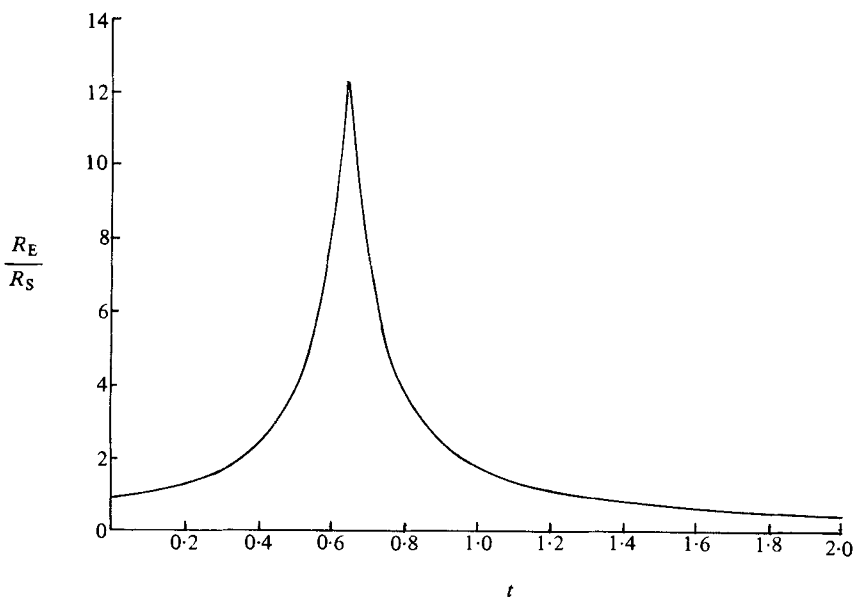

FIGURE 5. R_E/R_S versus time for $\eta = 0{\cdot}95$, $A = -1{\cdot}6$. An experimental trajectory is represented by a horizontal line at a starting condition defined by R_0/R_S.

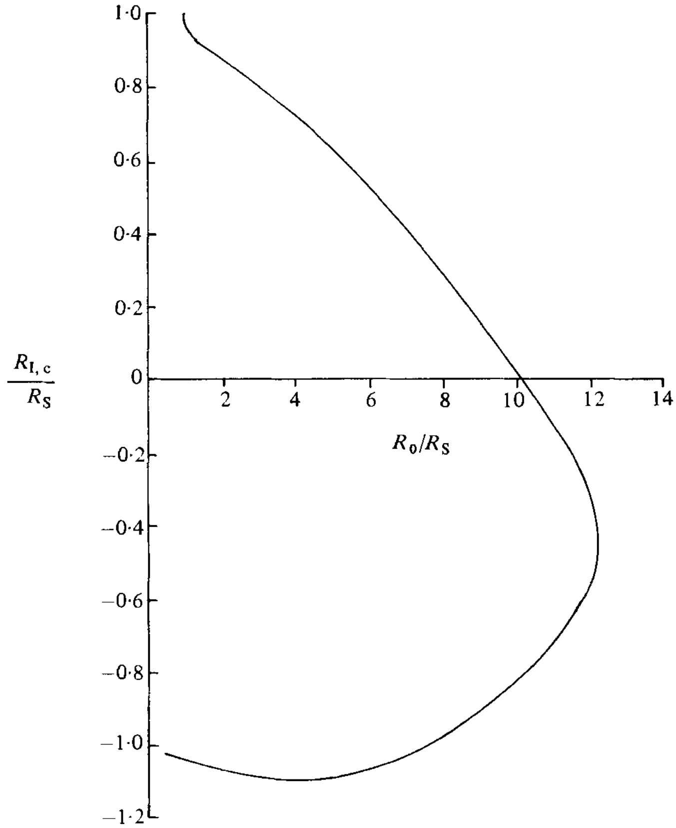

FIGURE 6. $R_{I,c}/R_S$ versus R_0/R_S for $\eta = 0{\cdot}95$, $A = -1{\cdot}6$.

decay appreciably, since the spike in the curve becomes quite narrow, especially for higher starting values. However, if one starts at a low-enough supercritical R_0, the disturbance may not only decay appreciably, but the flow will remain stable for quite some time following the reversal of cylinder direction. This can be seen in figure 6, which shows $R_{I,c}/R_S$ versus R_0/R_S for the same parameters as figure 5. On this figure and the ones like it, an experimental trajectory is represented by a vertical line that originates from a point on a straight line of slope one emanating from the origin. For this case of $\eta = 0{\cdot}95$, $A = -1{\cdot}6$ we can see that initially stability will not be guaranteed (for $R_0/R_S > 1$) until R_I is well below R_S, e.g. 48 % below R_S for $R_0/R_S = 6$. This reduced region of guaranteed stability indicates that hysteresis may indeed be possible. On the other hand, once we enter the region of strong stability for this R_0, disturbances continue to experience exponential decay until the cylinder reverses direction and $|R_I|$ is 6 % *above* R_S. This enhancement in the opposite direction is maximized for this case at about 10 % above R_S for $R_0 \approx 4R_S$. This compares with a maximum enhancement of 4 % obtained for the corresponding positive value of A.

Calculations for the wide-gap problem ($\eta = 0{\cdot}5$) were performed for six different values of the deceleration constant A ($\pm 0{\cdot}4$, $\pm 1{\cdot}6$, $\pm 5{\cdot}0$), and are presented in terms of instantaneous and initial Reynolds numbers in figures 7 and 8. Figure 7 shows results for $A > 0$. The amount of enhancement increases with A and amounts to over 12 % for the case $A = 5{\cdot}0$. This value of A is well out of the range of applicability

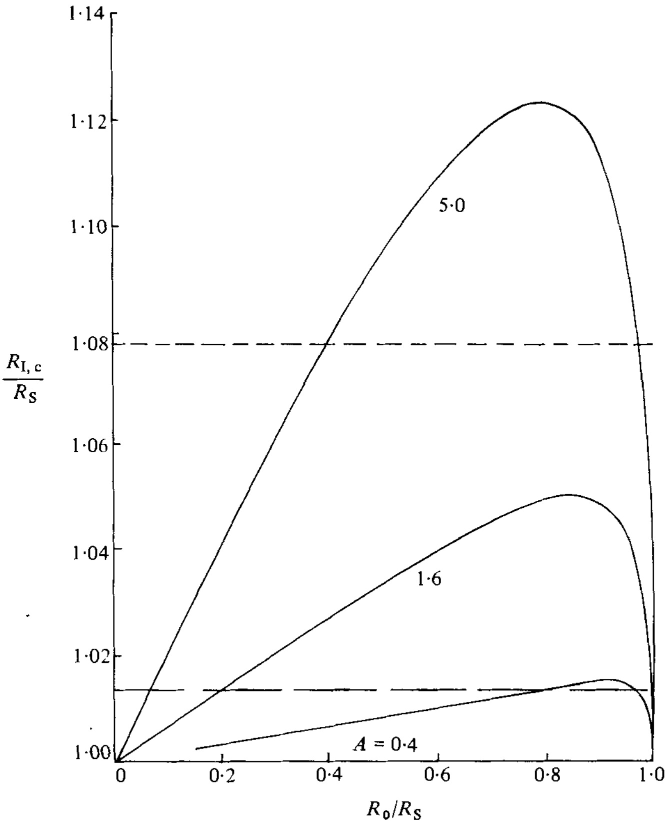

FIGURE 7. $R_{I,c}/R_S$ versus R_0/R_S for $\eta = 0{\cdot}5$ and $A > 0$: ——, steady-state linear limit; ---------, Eagles' enhanced limit for $A = 1{\cdot}6$.

of Eagles' analysis. This relationship between $R_{I,c}$ and A can be anticipated by imagining large values of A for which the change in ω becomes nearly impulsive. In this limit, a thin layer forms initially near the inner cylinder. In this thin shear layer, the stabilizing viscous forces are very high. This layer must thicken via diffusion before the destabilizing centrifugal forces can dominate, leading to instability. During this period, however, the inner-cylinder angular velocity can reach very high instantaneous values, resulting in stability being guaranteed for high values of R_I.

Also shown on figure 7 is the steady-state linear-theory limit for $\eta = 0{\cdot}5$. Notice that the value of $R_{I,c}$ at the point of maximum enhancement lies above this linear limit for all three values of A considered. Notice also that the point of maximum enhancement shifts to lower values of R_0 for higher values of A. For $A = 1{\cdot}6$, the enhancement is about 5 %, compared with 4 % obtained for $\eta = 0{\cdot}95$ with the same value of A. Eagles, on the other hand, predicts roughly 6 % enhancement of the linear limit for $\eta = 0{\cdot}5$ compared with 10 % for $\eta = 0{\cdot}95$. Eagles' enhanced value for $A = 1{\cdot}6$ is also plotted on figure 7 as a horizontal line. The present results are consistent with that of Eagles in that they lie below the linear-theory result as they must.

Results for $\eta = 0{\cdot}5$ and $A < 0$ are shown on figure 8. Semilogarithmic co-ordinates have been used to display all the results on one graph. For all three cases considered, the amount of enhancement (~ 40 %) following direction reversal is roughly the same. However, the starting values R_0 beyond which stability is never guaranteed (the

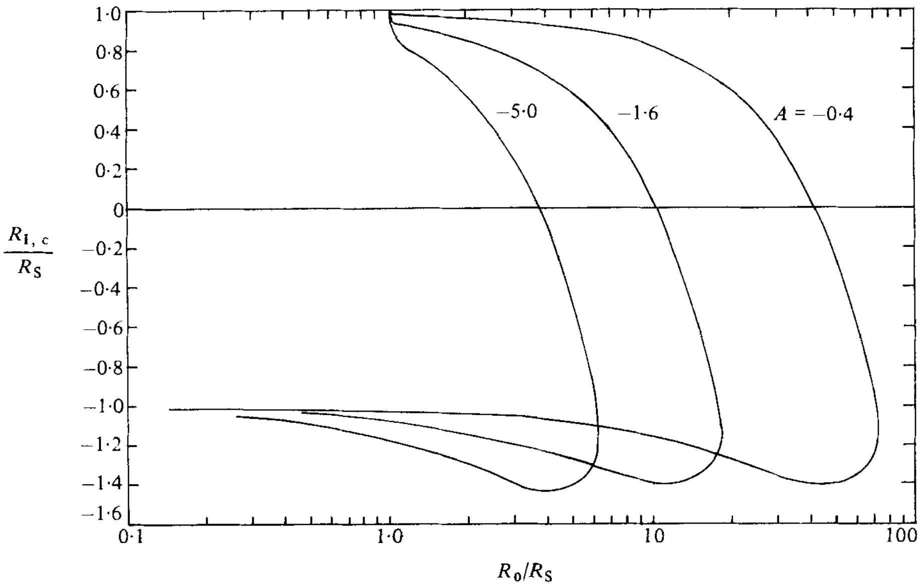

FIGURE 8. $R_{I,c}/R_S$ versus R_0/R_S for $\eta = 0{\cdot}5$ and $A < 0$.

rightmost point of each curve) decrease sharply with decreasing A. Also shown is the fact that larger negative values of A contribute to a reduced region of stability for early times. Hysteresis, therefore, appears more likely for more negative A. A comparison of the $A = -1{\cdot}6$ case with the corresponding case for $\eta = 0{\cdot}95$ (figure 6) shows a larger region of stability (in terms of both the maximum R_0 and maximum $|R_{I,c}|$) for the wide-gap case.

The behaviour of the critical wavenumber k_c, as a function of the acceleration A is of some interest, although it may not correspond to a dynamically admissible disturbance since ours is an energy-theory analysis. Chen *et al.* (1973) have determined critical wavelengths, both experimentally and numerically, for a similar problem in which the final inner-cylinder angular velocity was attained via constant acceleration from rest. In some of their cases, onset was noted after the final angular velocity had been reached. They observed a decrease in vortex wavelength with an increase in inner-cylinder acceleration. Figure 9 is a plot of k_c versus R_0/R_S for the three different values of A considered for $\eta = 0{\cdot}5$. These are plotted for selected points and were determined to three significant figures, which explains the choppy nature of the data. The trend, however, is clear: for a fixed starting condition R_0/R_S, the critical wavenumber increases slightly with increasing A, in qualitative agreement with the results of Chen *et al.*

These results clearly demonstrate the utility of energy-stability theory as applied to flows whose basic states vary with time. Unlike linear theory, energy theory is applicable to flows that are not necessarily slowly varying (or time-periodic). In fact, as mentioned in §1, for some of these flows energy theory is the only approach presently available, aside from direct numerical simulation. Energy theory provides a guarantee of stability against disturbances of arbitrary amplitude, which, in addition to being rigorously correct, is physically interesting. The present results indicate the possibility of hysteresis, especially for large values of inner cylinder acceleration/deceleration. Recent experiments by Park, Crawford & Donnelly (1981)

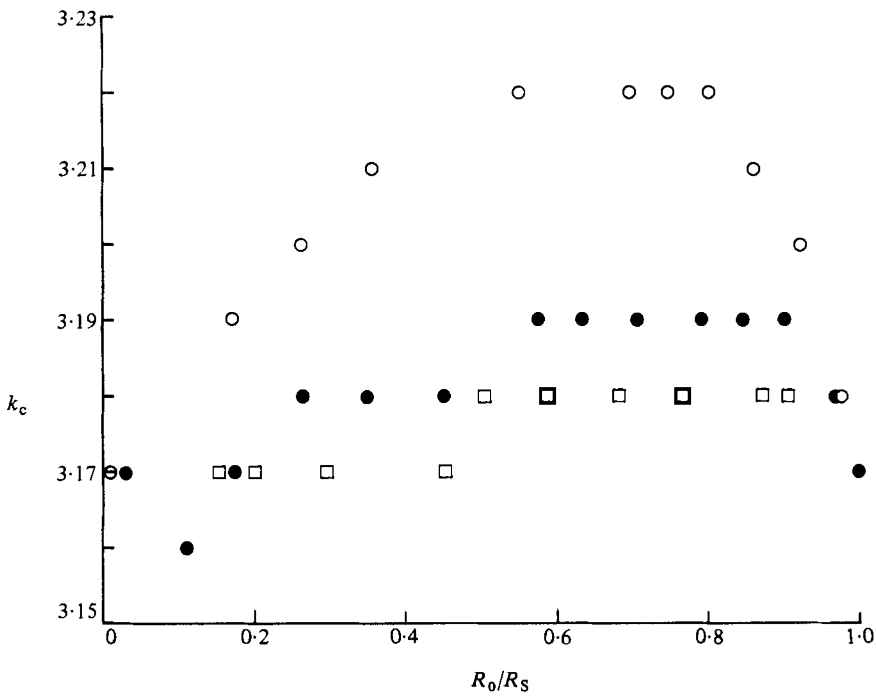

FIGURE 9. Variation of critical wavenumber with R_0/R_S and A: □, $A = 0{\cdot}4$; ●, 1·6; ○, 5·0.

have examined the effects of both cylinder acceleration and apparatus aspect ratio on the transition to Taylor-vortex flow. While a direct comparison with these results is impossible because of differing radius ratios and the fact that their initial conditions are not stated explicitly (i.e. whether each experiment was begun from a state of steady Couette flow and, if so, the corresponding value of R_0), the results of these experiments are in qualitative agreement with the present results; there is a definite hysteresis effect whose magnitude increases with increasing cylinder acceleration/deceleration. Finally, the results of the energy-theory calculations are not necessarily conservative. In particular, for the problem treated here with $\eta = 0{\cdot}5$ and $A > 0$, the results of energy and linear theories are sufficiently close that they can be used in the design of experiments.

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Appendix

The basic-state azimuthal velocity $V(r,t)$ is given in dimensionless form by

$$V(r,t) = \phi(r)\frac{\eta}{1-\eta}(1+At) + \sum_{i=1}^{\infty} U_1(\alpha_i r)\left\{\exp(-\alpha_i^2 t)\left[C_i + A_i\frac{\eta}{1-\eta}\left(1-\frac{A}{\alpha_i^2}\right)\right] + \frac{A_i A}{\alpha_i^2}\frac{\eta}{1-\eta}\right\},$$

where

$$\phi(r) = \frac{-\eta}{1+\eta}r + \frac{\eta}{(1-\eta^2)(1-\eta)}\frac{1}{r},$$

$$A = \frac{\tilde{A}(b-a)^2}{\nu},$$

$$U_k(\alpha_i r) = Y_1\left(\frac{\alpha_i \eta}{1-\eta}\right)J_k(\alpha_i r) - J_1\left(\frac{\alpha_i\eta}{1-\eta}\right)Y_k(\alpha_i r),$$

$$C_i = \frac{\eta}{\alpha_i Q_i(1-\eta^2)(1-\eta)}\left[\eta U_0\left(\frac{\alpha_i\eta}{1-\eta}\right) + U_0\left(\frac{\alpha_i}{1-\eta}\right)\right],$$

$$Q_i = \frac{1}{2(1-\eta)^2}\left[U_0^2\left(\frac{\alpha_i}{1-\eta}\right) - \eta^2 U_0^2\left(\frac{\alpha_i\eta}{1-\eta}\right)\right],$$

$$A_i = \frac{-\eta U_0\left(\frac{\alpha_i\eta}{1-\eta}\right)}{(1-\eta)\alpha_i Q_i}.$$

In the preceding, J_k and Y_k are the usual Bessel functions of the first and second kind respectively, and α_i are the roots of the equation

$$U_1\left(\frac{\alpha}{1-\eta}\right) = 0.$$

For any given η, the roots α_i and quantities C_i and A_i are constants and need only be computed once.

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